Large-N Analysis of Three Dimensional Nonlinear Sigma Models

Kiyoshi Higashijima [1*], **Etsuko Itou**[2†] **and Makoto Tsuzuki**[3‡]

[1]*Department of Physics, Graduate School of Science, Osaka University,*

Toyonaka, Osaka 560-0043, Japan

[2]*Institute of Theoretical Physics Kanazawa University*

Kakumamachi Kanazawa 920-1192, Japan

[3]*Toshiba Corporation Semiconductor Company*

Kawasaki,Kanagawa 212-8583,Japan

Abstract

Non-perturbative renormalization group approach suggests that a large class of nonlinear sigma models are renormalizable in three dimensional space-time, while they are non-renormalizable in perturbation theory. $\mathcal{N} = 2$ supersymmetric nonlinear sigma models whose target spaces are Einstein-Kähler manifolds with positive scalar curvature belongs to this class. hermitian symmetric spaces, being homogeneous, are specially simple examples of these manifolds. To find an independent evidence of the nonperturbative renormalizability of these models, the large N method, another nonperturbative method, is applied to 3-dimensional $\mathcal{N} = 2$ supersymmetric nonlinear sigma models on the target spaces $CP^{N-1} = SU(N)/[SU(N-1) \times U(1)]$ and $Q^{N-2} = SO(N)/[SO(N-2) \times SO(2)]$, two typical examples of hermitian symmetric spaces.

We find that β functions in these models agree with the results of the nonperturbative renormalization group approach in the next-to-leading order of $1/N$ expansion, and have nontrivial UV fixed points. The β function of the Q^{N-2} model receives a nonzero correction in the next-to-leading order of the $1/N$ expansion.

We also investigate the phase structures of our models. The CP^{N-1} model has two phases; $SU(N)$ symmetric and asymmetric phase. The Q^{N-2} model has three phases; Chern-Simons, Higgs and $SO(N)$ broken phases. In the Chern-Simons and Higgs phase, $SO(N)$ symmetry remains unbroken and all dynamical fields becomes massive. An auxiliary gauge field also acquires mass, through an induced Chern-Simons term in the Chern-Simons phase, and through the vacuum expectation value of a di-quark bound state in the Higgs phase.

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[*]E-mail: `higashij@phys.sci.osaka-u.ac.jp`

[†]E-mail: `itou@hep.s.kanazawa-u.ac.jp`

[‡]E-mail: `makoto.tsuzuki@toshiba.co.jp`

1 Introduction

Non-perturbative analyses reveal non-trivial features of quantum field theories which cannot be found within the perturbative analysis. One of the powerful non-perturbative method is the Wilsonian renormalization group (WRG) approach [1, 2, 3, 4]. In [5], WRG method was applied to three dimensional $\mathcal{N} = 2$ supersymmetric nonlinear sigma models (NLσM). Three dimensional NLσMs are perturbatively non-renormalizable according to the power counting. In the WRG approach, the renormalizability of NLσMs is equivalent to the existence of a nontrivial continuum limit, $\Lambda \to \infty$. When the ultraviolet (UV) cutoff Λ tends to infinity, we have to fine tune the coupling constant to the critical value at the UV fixed point, so as to keep the observable quantities finite. Therefore, it is important to show the existence of the UV fixed point without using the perturbation theory. [4] One of the main results in the paper [5] is the existence of non-trivial ultraviolet (UV) fixed point in the NLσMs whose target space are Einstein-Kähler manifolds with positive scalar curvature. This result implies that these models are renormalizable in the WRG approach. In the WRG method, the renormalization group equation is an exact equation for the most general effective action which includes an infinite number of coupling constants. To make this equation more tractable, we usually expand the effective action in powers of derivatives and retain the first few terms [6]. In the WRG analysis in [5], the most general effective action is approximated by truncating at the second order of derivative expansion. Since it is difficult to justify the validity of this approximation directly, we apply another powerful non-perturbative method, the large N method, to three dimensional $\mathcal{N} = 2$ supersymmetric nonlinear sigma models (NLσM) to show the existence of UV fixed point.

The $1/N$ expansion method has been applied to show the nonperturbative renormalizability of the three dimensional Gross-Neveu model [7, 8, 9, 10]. and NLσM with $O(N)$ symmetry [11, 12]. In NLσM, however, quadratic terms of an auxiliary field, introduced as a lagrange multiplier field, appears as counter terms and destroy the renormalizability in the strict sense. Three dimensional $O(N)$ symmetric NLσM with $\mathcal{N} = 1$ supersymmetry has been shown to be renormalizable in the large N method because of the supersymmetry [13]. The β function of this model is known to have a nontrivial UV fixed point. NLσM on the complex projective manifold $\mathbf{C}P^{N-1}$ has been formulated by introducing a auxiliary gauge field [14, 15, 16]. If we impose the gauge invariance to eliminate the possible digergent graphs, $\mathbf{C}P^{N-1}$ model is renormalizable in the large N method. Inami, Saito and Yamamoto have explicitly shown that $\mathcal{N} = 2$ supersymmetricNLσM on $\mathbf{C}P^{N-1}$ has no next-to-leading order contribution to the β function in the $1/N$ expansion [17], confirming the previous argument [18, 19].

In the large N method, we introduce N copies of dynamical fields and appropriate auxiliary fields to linearize the lagrangian of the dynamical fields, so that these dynamical fields can be

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[4]$\mathcal{N} = 2$ supersymmetry in three dimensions requires the target manifolds of NLσMs are Kähler manifolds.

integrated out in the path integral formalism [20]. Auxiliary field formulation of supersymmetric NLσMs has been found for a special class of Kähler-Einstein manifolds called hermitian symmetric spaces [16]. Hermitian symmetric spaces are homogeneous coset spaces with Kähler structure. Among the hermitian symmetric spaces, $\mathbf{C}P^{N-1} = SU(N)/[SU(N-1) \times U(1)]$ and $Q^{N-2} = SO(N)/[SO(N-2) \times SO(2)]$ are suitable for large N analysis. The β function for CP^{N-1} model agrees with the result of the WRG analysis. However, the WRG result suggests that the β function for the other $\mathcal{N} = 2$ supersymmetric NLσMs have next-to-leading order corrections in the $1/N$ expansion.

In this paper, we investigate the Q^{N-2} model, which is formulated explicitly with auxiliary fields in paper [21]. The β function of Q^{N-2} model obtained by the WRG analysis suggests that there is a next-to-leading order correction for the β function in $1/N$ expansion. Therefore, we explicitly calculate the β function of the Q^{N-2} model. Furthermore we study the phase structure of the CP^{N-1} and the Q^{N-2} models. The $\mathbf{C}P^{N-1}$ model is known to have two phases; $SU(N)$ symmetric and asymmetric phase. In the symmetric phase, all dynamical fields acquire a mass, while they play the role of the massless Nambu-Goldstone bosons and their superpartners in the broken phase. The Q^{N-2} model turns out to have three phases; Chern-Simons, Higgs and $SO(N)$ broken phases. In the Chern-Simons and Higgs phase, $SO(N)$ symmetry remains unbroken and all dynamical fields becomes massive, while there are $(N-2)$ massless Nambu-Goldstone bosons and their superpartners in the $SO(N)$ broken phase. In the Chern-Simons phase, a auxiliary gauge fields becomes massive through an induced Chern-Simons term, while in Higgs phase the gauge field acquire the mass by the Higgs mechanism through the vacuum expectation value of a di-quark bound state.

This paper is organized as follows. First, we will reconfirm the β function of $\mathbf{C}P^{N-1}$ model has no next-to-leading contributions in $1/N$ expansion, and will compare the β function with the WRG result in §.2. We will also investigate the phase structures of $\mathbf{C}P^{N-1}$ in this section. Next we will investigate the phase structure and the β function of the Q^{N-2} model in §.3.

2   The $\mathbf{C}P^{N-1}$ model

In this section, we review the large-N analysis of the $\mathbf{C}P^{N-1}$ model. The β function of this model has no next-to-leading order correction because of $\mathcal{N} = 2$ supersymmetry [17].

2.1   The auxiliary field formulation of $\mathbf{C}P^{N-1}$ model

Let us introduce chiral superfields $\Phi^i(x,\theta)$ $(i = 1, 2, \cdots, N)$ belonging to a fundamental representation of $G = SU(N)$, the isometry group of $\mathbf{C}P^{N-1}$. We also introduce $U(1)$ gauge symmetry

$$\Phi(x,\theta) \longrightarrow \Phi'(x,\theta) = e^{i\Lambda(x,\theta)}\Phi(x,\theta) \tag{2.1}$$

to require that $\Phi(x,\theta)$ and $\Phi'(x,\theta)$ are physically indistinguishable. With a complex chiral super-field, $e^{i\Lambda(x,\theta)}$ is an arbitrary complex number. $U(1)$ gauge symmetry is thus complexified to $U(1)^{\mathbf{C}}$. The identification $\Phi \sim \Phi'$ defines the complex projective space $\mathbf{C}P^{N-1}$. In order to impose local $U(1)$ gauge symmetry, we have to introduce a $U(1)$ gauge field $V(x,\theta,\bar{\theta})$ with the transformation property $e^{-V} \longrightarrow e^{-V}e^{-i\Lambda+i\Lambda^*}$. $V(\theta,\bar{\theta},x)$ is a real scalar superfield[22], and defined by dimensional reduction from 4-dimensional $\mathcal{N} = 1$ to 3-dimensions. Then the lagrangian with a local $U(1)$ gauge symmetry is given by

$$\mathcal{L} = \int d^4\theta(\Phi^i\Phi^{\dagger\bar{i}}e^{-V} + cV), \tag{2.2}$$

where the last term V is called the Fayet-Illiopoulos D-term. In this model, the gauge field $V(x,\theta,\bar{\theta})$ is an auxiliary superfield without kinetic term. The Kähler potential $K(\Phi,\Phi^*)$ is obtained by eliminating V using the equation of motion for V

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}K(\Phi,\Phi^*) = c\int d^2\theta d^2\bar{\theta}\log(\vec{\Phi}^* \cdot \vec{\Phi}). \tag{2.3}$$

This Kähler potential reduces to the standard Fubini-Study metric of $\mathbf{C}P^{N-1}$

$$K(\phi,\phi^*) = c\log\left(1 + \sum_{i=1}^{N-1}\Phi^{i*}\Phi^i\right) \tag{2.4}$$

by a choice of gauge fixing

$$\Phi^N(x,\theta) = 1. \tag{2.5}$$

The global symmetry $G = SU(N)$, the isometry of the target space $\mathbf{C}P^{N-1}$, is linearly realized on our Φ^i fields and our lagrangian (2.2) with auxiliary field V is manifestly invariant under G. The gauge fixing condition (2.5) is not invariant under $G = SU(N)$ and we have to perform an appropriate gauge transformation simultaneously to compensate the change of Φ^N caused by the $SU(N)$ transformation. Therefore the global symmetry $G = SU(N)$ is nonlinearly realized in the gauge fixed theory. In this sense, our lagrangian (2.2) use the linear realization of G in contrast to the nonlinear lagrangian (2.4) in terms of the Kähler potential which use the nonlinear realization of G.

Instead of fixing the gauge to eliminate one component of chiral superfield, we can eliminate the chiral and and anti-chiral components in the gauge superfield $V(x,\theta,\bar{\theta})$. In our discussion below, we employ this Wess-Zumino gauge. Then the gauge superfield is written by component fields:

$$V = \bar{\theta}\gamma^\mu\theta v_\mu + \bar{\theta}\theta M + \frac{1}{2}\theta^2\bar{\theta}\lambda + \frac{1}{2}\bar{\theta}^2\bar{\lambda}\theta + \frac{1}{4}\theta^2\bar{\theta}^2 D,$$

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where v_μ is a gauge field of three dimensional theory and the scalar field M corresponds to the fourth component of 4-dimensional vector field. Note that the real part of the scalar component of the gauge transformation $\Lambda(x,\theta)$ is not fixed yet with this gauge choice. We have to fix this residual gauge freedom to calculate the $1/N$ correction where this auxiliary gauge field begins to propagate. In order to make the Lagrangian of order N in the large N limit, we take the coefficient Fayet-Illiopoulous D-term $c = N/g^2$ and keep g^2 fixed when we take the limit of $N \to \infty$. With this gauge choice, the global $SU(N)$ symmetry coming from the isometry of the target manifold is linearly realized, while the supersymmetry is realized non-linearly. Furthermore, the action is invariant under $U(1)$ gauged symmetry generated by the real part of the gauge function $\Lambda(x,\theta)$, with the assignment of the $U(1)$ charge

$$[\Phi] = 1, \quad [\Phi^\dagger] = -1. \tag{2.6}$$

Integrating out the Grassmann coordinates θ and $\bar\theta$, the action (2.2) is written by using the component fields:

$$
\begin{aligned}
\mathcal{L} &= \partial_\mu \varphi^{*i} \partial^\mu \varphi^i + i\bar\psi^i \slashed{\partial} \psi^i + F^i F^{i*} - [i(\varphi^{*i}\partial_\mu\varphi^i - \varphi^i \partial_\mu \varphi^{i*}) + \bar\psi^i \gamma_\mu \psi^i]v^\mu + v^\mu v_\mu \varphi^{*i}\varphi^i \\
&\quad - M^2 \varphi^{*i}\varphi^i - M\bar\psi^i \psi^i - D\varphi^{*i}\varphi^i + \frac{N}{g^2}D + (\varphi^i \bar\psi^i \lambda + \varphi^{i*}\bar\lambda \psi^i)
\end{aligned} \tag{2.7}
$$

Since the gauge superfield V or its component fields do not have kinetic terms, they are auxiliary fields and do not propagate in the tree approximation. If we eliminate all auxiliary fields using their equations of motion, we obtain the constraints

$$
\varphi^i \varphi^{*i} = \frac{N}{g^2}, \quad M = -\frac{g^2}{2N}\bar\psi^i \psi^i,
$$
$$
v^\mu = \frac{g^2}{2N}\left[i(\varphi^{*i}\partial_\mu\varphi^i - \varphi^i \partial_\mu \varphi^{i*}) + \bar\psi^i \gamma^\mu \psi^i\right], \tag{2.8}
$$
$$
\varphi^i \bar\psi^i = \varphi^{*i}\psi^i = 0.
$$

The first equation means fields φ^i are constrained on the $(2N - 1)$ dimensional sphere S^{2N-1}. Furthermore, the gauge transformation of gauge field v_μ eliminates a common phase of φ^i. Thus, the target manifold reduces to the complex projective space $\mathbf{C}P^{N-1}$. The last equation of (2.8) implies that fermion resides on the tangent space of $\mathbf{C}P^{N-1}$.

2.2 Phase structure of $\mathbf{C}P^{N-1}$ model

To investigate the phase structure, let us calculate the effective potential in the leading order of the $1/N$ expansion. The partition function of this model can be written as

$$Z = \int D\Phi^i D\Phi^{\dagger i} DV e^{i\int d^3x \mathcal{L}}, \tag{2.9}$$

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where the Lagrangian $\mathcal{L}$ is given in (2.2) and the measure DV includes the gauge fixing term of the remaining gauge freedom in the Wess-Zumino gauge. In order to calculate the effective potential, the vacuum energy when $\langle \varphi(x) \rangle = \tilde{\varphi}$ is kept fixed, we divide the dynamical field into the vacuum expectation value and the fluctuation, $\varphi^i = \tilde{\varphi}^i + \varphi'^i$. The fluctuation field has to follow a constraint [21, 23]

$$\int d^3 x \varphi'^i(x) = 0, \tag{2.10}$$

which forbids the appearance of tadpole or one particle reducible graphs in the effective potential. Integrating over the fluctuation field φ'^i, we obtain the effective action:

$$Z = \int DV e^{iS_{eff}}, \tag{2.11}$$

$$S_{eff} = -\frac{N}{i} \text{Tr} \ln(\nabla_B + \bar{\lambda} \nabla_F^{-1} \lambda) + \frac{N}{i} \text{Tr} \ln \nabla_F \tag{2.12}$$

$$+ \int d^3 x \left(\frac{N}{g^2} D - \tilde{\varphi}^{*i}(M^2 + D)\tilde{\varphi}^i + (\lambda\text{-dependent terms}) \right).$$

Here we use the following notation

$$\nabla_B = D^\mu D_\mu + (M^2 + D), \quad \nabla_F = i\gamma^\mu D_\mu - M, \quad D_\mu = \partial_\mu + iv_\mu.$$

Assuming the Lorentz invariance of the vacuum, we take the vacuum expectation value of each auxiliary field as follow:

$$\langle M(x) \rangle = M_0, \quad \langle D(x) \rangle = D_0, \quad \langle \text{others} \rangle = 0.$$

We evaluate the momentum integration by introducing the ultraviolet cutoff Λ, then we obtain the effective potential in the leading order of the $1/N$ expansion ($\not{k} = \gamma^\mu k_\mu$).

$$\frac{V_{\text{eff}}}{N} = \int^\Lambda \frac{d^3 k}{(2\pi)^3} \ln(-k^2 + M_0^2 + D_0) - \int^\Lambda \frac{d^3 k}{(2\pi)^3} \text{tr} \ln(\not{k} - M_0)$$

$$+ \frac{1}{N} \tilde{\varphi}^{i*}(M_0^2 + D_0)\tilde{\varphi}^i - \frac{1}{g^2} D_0$$

$$= -\frac{1}{6\pi}|M_0^2 + D_0|^{\frac{3}{2}} + \frac{1}{6\pi}|M_0|^3 + \frac{1}{N}(M_0^2 + D_0)|\tilde{\varphi}^i|^2 + (\frac{\Lambda}{2\pi^2} - \frac{1}{g^2})D_0.$$

$$= -\frac{1}{6\pi}|M_0^2 + D_0|^{\frac{3}{2}} + \frac{1}{6\pi}|M_0|^3 + \frac{1}{N}(M_0^2 + D_0)|\tilde{\varphi}^i|^2 + \frac{m}{4\pi}D_0, \tag{2.13}$$

where we defined a renormalized coupling constant g_R to absorb the linear divergence that appears in the coefficient of D_0

$$\frac{\mu}{g_R^2} = \frac{1}{g^2} - \frac{1}{2\pi^2}\Lambda + \frac{\mu}{4\pi}. \tag{2.14}$$

with an arbitary finite mass scale μ called the renormalization point. We have also defined a renormalization group invariant mass m given by

$$m \equiv \mu(1 - \frac{4\pi}{g_R^2}). \tag{2.15}$$

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To evaluate the effective potential in the presence of $\tilde\varphi, M_0, D_0$, we have to perform the path integration over the nonzero mode of auxiliary fields in (2.11). When N is very large, these path integration can be done by the saddle point method since the S_{eff} is of order N. In the leading order of $1/N$ expansion, the effective potential is given by the value of S_{eff} at the saddle point. Assuming the saddle point is located at the translationally invariant configuration, we can neglect the non-zero mode of the auxiliary fields in the large N limit. The saddle point condition is, therefore, equivalent to the the stationary condition of the effective potential (2.13):

$$\frac{1}{N}\frac{\partial V_{\text{eff}}}{\partial M_0} = -2M_0\left(\frac{\epsilon}{4\pi}|M_0^2 + D_0|^{\frac{1}{2}} - \frac{1}{4\pi}|M_0| - \frac{1}{N}|\tilde\varphi^i|^2\right) = 0, \tag{2.16}$$

$$\frac{1}{N}\frac{\partial V_{\text{eff}}}{\partial D_0} = -\frac{\epsilon}{4\pi}|M_0^2 + D_0|^{\frac{1}{2}} + \frac{1}{N}|\tilde\varphi^i|^2 + \frac{m}{4\pi} = 0 \tag{2.17}$$

where $\epsilon = \text{sgn}(M_0^2 + D_0)$. These two conditions fixes the value of M_0 at the saddle point

$$|M_0| = 0 \quad \text{or} \quad m. \tag{2.18}$$

We will discuss these two cases separately.

$|M_0| = m$ **case:** This case is possible only when $m \geq 0$. By solving (2.17) for $|D_0 + m^2|$ and substituting it back to the effective potential (2.13), we obtain the vacuum energy when $\tilde\varphi$ is kept fixed

$$V_{\text{eff}}(\tilde\varphi) = \frac{N}{12\pi}\left(\left|\frac{4\pi}{N}|\tilde\varphi^i|^2 + m\right|^3 - m^3\right) \tag{2.19}$$

Because we are assuming $m \geq 0$, the minimum of this vacuum energy is located at $\tilde\varphi^i = 0$, (2.17) implies $D_0 = 0$. Since $\tilde\varphi$ and D_0 are the order parameter of $SU(N)$ and supersymmetry respectively, neither $SU(N)$ nor supersymmetry is broken in this case.

$M_0 = 0$ **case:** By solving (2.17) for $|D_0|$ and substituting it back to the effective potential (2.13), we obtain the vacuum energy

$$V_{\text{eff}}(\tilde\varphi) = \frac{N}{12\pi}\left|\frac{4\pi}{N}|\tilde\varphi^i|^2 + m\right|^3 \tag{2.20}$$

When $m \geq 0$, the minimum located at $\tilde\varphi = 0$ has a higher vacuum energy than the previous vacuum, and does not correspond to the ground state. On the other hand, when $m < 0$, the minimum is at

$$|\tilde\varphi^i|^2 = \frac{N}{4\pi}|m|, \tag{2.21}$$

then (2.17) implies $D_0 = 0$, namely supersymmetry is unbroken while $SU(N)$ symmetry is spontaneously broken in this case.

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Thus we found two vacuum, both of them are supersymmetric. In the following, we will discuss these two phases in detail. We first discuss the mass spectrum of dynamical fields by substututing the vacuum expectation values in the lagrangian.Then we evaluate the two point functions of the auxiliary fields to discuss their masses.

1. $SU(N)$ symmetric phases ($|\tilde{\varphi}^i|^2 = 0, M_0 = m$)

   By replacing M by $m + M'$ in the lagrangian (2.7):

   $$
   \begin{aligned}
   \mathcal{L} = {} & -\varphi^{*i}(\partial_\mu \partial^\mu + m^2)\varphi^i + \bar{\psi}^i(i\slashed{\partial} - m)\psi^i + F^i F^{*i} \\
   & -[i(\varphi^{*i}\partial_\mu \varphi^i - \varphi^i \partial_\mu \varphi^{*i}) + \bar{\psi}^i \gamma_\mu \psi^i]v^\mu + v^\mu v_\mu \varphi^{*i}\varphi^i \\
   & -(2mM' + M'^2)\varphi^{*i}\varphi^i - M'\bar{\psi}^i\psi^i - D\varphi^{*i}\varphi^i + \frac{N}{g^2}D + (\varphi^i \bar{\psi}^i \lambda + \varphi^{*i}\bar{\lambda}\psi^i),
   \end{aligned}
   \tag{2.22}
   $$

   we find that N scalar and spinor fields have the same mass m by the vacuum expectation value of M_0, and global SU(N) and local $U(1)$ gauge symmetry are unbroken in this phase. Integrating out the dynamical field, we obtain the effective action (2.12) describing the dynamics of auxiliary fields in this phase.

   $$
   \begin{aligned}
   S_{eff} = {} & -\frac{N}{i}\text{Tr}\ln\{\partial_\mu \partial^\mu + i(v^\mu \partial_\mu + \partial^\mu v_\mu) - v^\mu v_\mu + \bar{\lambda}[i\slashed{\partial} - \slashed{v} - (m + M')]^{-1}\lambda \\
   & +[(m + M')^2 + D]\} + \frac{N}{i}\text{Tr}\ln[i\slashed{\partial} - \slashed{v} - (m + M')] + \int dx \frac{N}{g^2}D.
   \end{aligned}
   \tag{2.23}
   $$

   When we expand this S_{eff} in the power series of fields, bilinear terms define the propagator of the auxiliary fields, and higher order terms define the interaction terms of auxiliary fields. Two point function of all auxiliary fields are summarized in Appendix.A.1, there we find the mass of the fields v_μ, M, λ is twice of that of the dynamical fields. These auxiliary field v_μ, M and λ represent the bound states of fermion–anti-fermion and fermion–boson as is shown in (2.8).

2. $SU(N)$ broken phases ($|\tilde{\varphi}^i|^2 = \frac{N}{4\pi}|m|, M_0 = 0$)

   Using $SU(N)$ symmetry, we choose the vacuum expectation value of $\tilde{\varphi}$ in the N-th direction: $\tilde{\varphi}^N = \sqrt{\frac{N}{4\pi}|m|}$. Substituting $\varphi^N = \sqrt{\frac{N}{4\pi}|m|} + \varphi'^N$ to (2.7), we obtain

   $$
   \begin{aligned}
   \mathcal{L} = {} & \mathcal{L}_{N-1} + \mathcal{L}_N \\
   \mathcal{L}_{N-1} = {} & \sum_{i=1}^{N-1}\left[-\varphi^{i*}(D_\mu D^\mu + M^2 + D)\varphi^i + \bar{\psi}^i(i\gamma^\mu D_\mu - M)\psi^i + F^i F^{i*} + (\varphi^i \bar{\psi}^i \lambda + \varphi^{i*}\bar{\lambda}\psi^i)\right] \\
   \mathcal{L}_N = {} & -\varphi'^{N*}(D_\mu D^\mu + M^2 + D)\varphi'^N + \bar{\psi}^N(i\gamma^\mu D_\mu - M)\psi^N + F^N F^{N*} + \frac{N}{g^2}D \\
   & +(\varphi'^N \bar{\psi}^N \lambda + \varphi'^{N*}\bar{\lambda}\psi^N) \\
   & +\sqrt{\frac{N}{4\pi}|m|}\left[-D(\varphi'^N + \varphi'^{N*}) - iv^\mu \partial_\mu(\varphi'^N - \varphi'^{N*}) + (\bar{\psi}^N \lambda + \bar{\lambda}\psi^N)\right] \\
   & -\sqrt{\frac{N}{4\pi}|m|}(M^2 - v^\mu v_\mu)(\varphi'^N + \varphi'^{N*}) - \frac{N}{4\pi}|m|(M^2 - v^\mu v_\mu)
   \end{aligned}
   \tag{2.24}
   $$

In this phase, global $SU(N)$ symmetry is broken down to $SU(N-1)$ and there are $N-1$ massless Nambu-Goldstone bosons φ^i $(i = 1, \cdots, N-1)$, and their superpartners ψ^i $(i = 1, \cdots, N-1)$. Furthermore, gauged $U(1)$ symmetry is also broken because the dynamical fields φ^N, carrying $U(1)$ charge, have a nonvanishing vacuum expectation value. The third line of $\mathcal{L}_N$ represents the mixing between φ'^N, ψ^N and auxiliary fields. Integrating over φ^i, ψ^i with $i = 1, \cdots, N$ in (2.11), we obtain the effective action for auxiliary fields:

$$
\begin{aligned}
S_{eff} &= -\frac{N-1}{i} \mathrm{Tr} \ln \{ \partial^\mu \partial_\mu + i(v^\mu \partial_\mu + \partial^\mu v_\mu) - v^\mu v_\mu + \bar{\lambda}[i\partial\!\!\!/ - \psi\!\!\!/ - M]^{-1}\lambda + [M^2 + D] \} \\
&\quad + \frac{N-1}{i} \mathrm{Tr} \ln[i\partial\!\!\!/ - \psi\!\!\!/ - M] + \int dx \frac{N}{g^2} D.
\end{aligned}
\tag{2.25}
$$

In order to obtain propagators of auxiliary fields we have to take into account the mixing between auxiliary fields and φ'^N, ψ^N. To diagonalize the propagators, we add the quadratic terms in (2.24) to the bilinear terms of S_{eff} to define $S_{eff}^{(2)}$

$$
\begin{aligned}
S_{eff}^{(2)} &= \int \frac{d^3 p}{(2\pi)^3} \left(\frac{1}{2} M(-p) \left(-\frac{N}{8}\sqrt{-p^2} - \frac{N}{2\pi}|m| \right) M(p) - \frac{1}{2} D(-p) \frac{N}{8\sqrt{-p^2}} D(p) \right. \\
&\quad - \frac{1}{2} v^\mu(-p) \left(\frac{N}{8} \frac{p^2 \eta_{\mu\nu} - p_\mu p_\nu}{\sqrt{-p^2}} - \frac{N}{2\pi}|m|\eta_{\mu\nu} \right) v^\nu(p) + \bar{\lambda}(-p) \frac{N\psi\!\!\!/}{16\sqrt{-p^2}} \lambda(p) \\
&\quad + \frac{1}{2} \varphi_R^N(-p) p^2 \varphi_R^N(p) + \frac{1}{2} \varphi_I^N(p) p^2 \varphi_I^N(p) + \bar{\psi}^N(-p) \psi\!\!\!/ \psi^N(p) \\
&\quad - \frac{1}{2}\sqrt{\frac{N}{2\pi}}|m|(\varphi_R^N D(p) + D(-p)\varphi_R^N) + \frac{1}{2}\sqrt{\frac{N}{2\pi}}|m|(\bar{\psi}^N(-p)\lambda(p) + \bar{\lambda}(-p)\psi^N(p)) \\
&\quad \left. - \frac{1}{2} ip^\mu \sqrt{\frac{N}{2\pi}}|m|(v_\mu(-p)\varphi_I^N(p) - \varphi_I^N(-p)v_\mu(p)) \right),
\end{aligned}
\tag{2.26}
$$

where $\varphi'^N = \frac{1}{\sqrt{2}}(\varphi_R^N + i\varphi_I^N)$. We immediately find that the propagator of the auxiliary field M has no pole. To solve the mixing between the gauge field v^μ and φ_I^N, we define a new vector field $U^\mu(p) = v^\mu(p) - ip^\mu(\sqrt{\frac{N}{2\pi}|m|})^{-1}\varphi_I^N$ and obtain its propagator:

$$
\Pi_{\mu\nu}^{-1} = \frac{1}{N} \frac{1}{-\frac{1}{8}\sqrt{-p^2} - \frac{1}{2\pi}|m|} \left(\eta_{\mu\nu} - \frac{2\pi}{|m|} \frac{p_\mu p_\nu}{8\sqrt{-p^2}} \right).
\tag{2.27}
$$

The singularity of this propagator is located at

$$
-\frac{1}{8}\sqrt{-p^2} - \frac{1}{2\pi}|m| = 0,
\tag{2.28}
$$

This equation has no solution in the physical sheet: $-\pi \leq \arg(-p^2) \leq \pi$. Similarly D, λ mix with φ_R^N, ψ^N, respectively, Then the eigen values of the field propagators are calculated as

follow:

$$\frac{1}{2}\begin{pmatrix} D(-p) & \varphi_R^N(-p) \end{pmatrix}\begin{pmatrix} \frac{N}{8\sqrt{-p^2}} & -\sqrt{\frac{N}{2\pi}}|m| \\ -\sqrt{\frac{N}{2\pi}}|m| & p^2 \end{pmatrix}\begin{pmatrix} D(p) \\ \varphi_R^N(p) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} D(-p) & \varphi_R^N(-p) \end{pmatrix} G^{-1}\begin{pmatrix} D(p) \\ \varphi_R^N(p) \end{pmatrix},$$

$$\begin{pmatrix} \bar{\lambda}(-p) & \bar{\psi}^N(-p) \end{pmatrix}\begin{pmatrix} \frac{N\slashed{p}}{16\sqrt{-p^2}} & \sqrt{\frac{N}{4\pi}}|m| \\ \sqrt{\frac{N}{4\pi}}|m| & \slashed{p} \end{pmatrix}\begin{pmatrix} \lambda(p) \\ \psi^N(p) \end{pmatrix} = \begin{pmatrix} \bar{\lambda}(-p) & \bar{\psi}^N(-p) \end{pmatrix} S^{-1}\begin{pmatrix} \lambda(p) \\ \psi^N(p) \end{pmatrix}.$$

$$(2.29)$$

The poles of the new propagators (G and S) are obtained by solving

$$\det G^{-1} = -\frac{N}{8}\sqrt{-p^2} - \frac{N}{2\pi}|m| = 0, \tag{2.30}$$

$$\det S^{-1} = -\frac{N}{16}\sqrt{-p^2} - \frac{N}{4\pi}|m| = 0. \tag{2.31}$$

However these equations have no solution in the physical Riemann sheet. Thus we find all auxiliary fields are not physical particles in the broken phase.

2.3　The β function of CP^{N-1}

To derive β function, we calculate the next-to-leading correction of the gap equation in the symmetric phase. The β function is determined by the UV behavior and we should be able to define a universal β function in both phases. Feynman rules in symmetric phase are summarized in Appendix A.1. In our notation, propagators of auxiliary field have the factor $1/N$.

The gap equation is given by equation of motion of the auxiliary field D.

$$\frac{N}{g^2} = \int \frac{d^3p}{(2\pi)^3}\langle \varphi^i(p)\varphi^{*i}(0)\rangle. \tag{2.32}$$

In the leading order of $1/N$, the gap equation is written as follow:

$$\frac{N}{g^2} = \int \frac{d^3p}{(2\pi)^3}\frac{iN}{p^2 - m^2}. \tag{2.33}$$

We renormalize the coupling constant to absorb the divergence using UV cutoff and obtain the leading order β function of $1/N$ expansion.

$$\beta(g_R) = -\frac{1}{8\pi}g_R^3 + \frac{1}{2}g_R \tag{2.34}$$

We calculate the next-to-leading Feynman diagrams, shown Fig.1, and find the next-to-leading correction of β function vanishes because of $\mathcal{N} = 2$ supersymmetry.

Finally, let's compare the result with the WRG result of CP^{N-1} case [5]:

$$\beta(\lambda) = -\frac{N}{4\pi^2}\lambda^3 + \frac{1}{2}\lambda. \tag{2.35}$$

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We define the 't Hooft coupling $g^2 \equiv N\lambda^2$, and find both β functions are same. This β function is shown in Fig.2. From the eq.(2.15), the region where the coupling constant is smaller than the critical point g_c corresponds to broken phase.
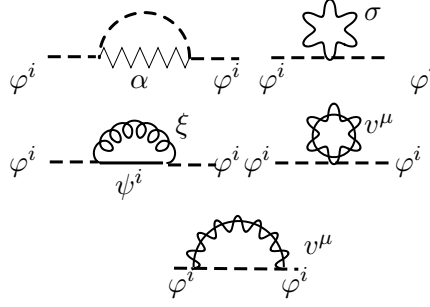

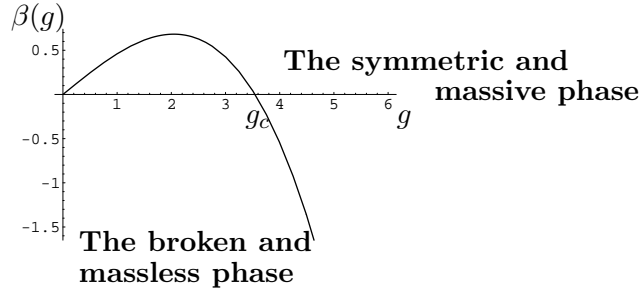Figure 1: The Feynman diagrams contributing to the next-to-leading order correction of φ propagator

Figure 2: The β function of the coupling constant g for the CP^{N-1} model.

3 The Q^{N-2} model

3.1 The theory and symmetry

The β function for the CP^{N-1} model has not next-to-leading order correction because of $\mathcal{N} = 2$ supersymmetry. The question is that all $\mathcal{N} = 2$ supersymmetric nonlinear sigma models have no next-to-leading corrections in terms of $1/N$ expansion. To investigate it, we consider the other $\mathcal{N} = 2$ supersymmtric nonlinear sigma model; the Q^{N-2} model. The Wilsonian renormalization group (WRG) approach for the Q^{N-2} model gives the following β function:

$$\beta(\lambda) = -\frac{N-2}{4\pi^2}\lambda^3 + \frac{1}{2}\lambda,$$

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This equation can be rewritten using 't Hooft coupling $g^2 = N\lambda^2$,

$$\beta(g) = -\frac{g^3}{4\pi^2}(1 - \frac{2}{N}) + \frac{1}{2}g. \tag{3.1}$$

This WRG result shows this model has the next-to-leading correction of β function in $1/N$ expansion.

Let's investigate Q^{N-2} model using $1/N$ expansion. This model is obtained from CP^{N-1} model with the $O(N)$ condition $\Phi^i\Phi^i = 0$. Then the Lagrangian has the two kinds of auxiliary fields as follow:

$$\mathcal{L} = \int d^4\theta(\Phi^i\Phi^{\dagger i}e^V - cV) + \frac{1}{2}\left(\int d^2\theta\Phi_0\Phi^i\Phi^i + \int d^2\bar{\theta}\Phi_0^\dagger\Phi^{\dagger i}\Phi^{\dagger i}\right), \tag{3.2}$$

where $V(\theta, \bar{\theta}, x)$ is $U(1)$ gauge superfield and $\Phi_0, \Phi_0^\dagger$ are chiral- and antichiral- superfields respectively. Components of the new auxiliary field Φ_0 is defined by

$$\Phi_0(y) = A_0(y) + \sqrt{2}\theta\psi_0^i(y) + \theta\theta F_0(y).$$

This auxiliary superfield is an $O(N)$ singlet and has gauged nonzero $U(1)$ charge; $[\Phi_0] = -2$ and $[\Phi_0^\dagger] = 2$. The dynamical fields Φ^i is a vector representation of global $O(N)$ symmetry and has the gauged $U(1)$ charge (2.6). Then the Q^{N-2} model has global $SO(N)$ symmetry coming from the isometry of the target manifold and the gauged $U(1)$ symmetry.

Integrated out the Grassmann coordinates θ and $\bar{\theta}$, we obtain the Q^{N-2} model Lagrangian in component fields:

$$
\begin{aligned}
\mathcal{L} = {} & \partial_\mu\varphi^{*i}\partial^\mu\varphi^i + i\bar{\psi}\partial_\mu\gamma^\mu\psi^i + F^iF^{*i} - [i(\varphi^{*i}\partial_\mu\varphi^i - \varphi^i\partial_\mu\varphi^{*i}) - \bar{\psi}^i\gamma_\mu\psi^i]v^\mu + v^\mu v_\mu\varphi^{*i}\varphi^i \\
& - M^2\varphi^{*i}\varphi^i - M\bar{\psi}^i\psi^i - D\varphi^{*i}\varphi^i + \frac{N}{g^2}D + (\varphi^i\bar{\psi}^i\lambda^c + \varphi^{*i}\lambda^c\psi^i) \\
& + \frac{1}{2}(F_0\varphi^i\varphi^i + F_0^*\varphi^{*i}\varphi^{*i}) - (\bar{\psi}_0^c\psi^i\varphi^i + \bar{\psi}^i\psi_0^c\varphi^{*i}) \\
& + A_0(\varphi^iF^i - \frac{1}{2}\bar{\psi}^{ci}\psi^i) + A_0^*(\varphi^{*i}F^{*i} - \frac{1}{2}\bar{\psi}^i\psi^{ci}). \tag{3.3}
\end{aligned}
$$

If we eliminate auxiliary fields using their equation of motion, we obtain the constraint eqs.(2.8) and additional following equations.

$$\varphi^i\varphi^i = \varphi^{*i}\varphi^{*i} = 0, \quad \varphi^i\psi^i = \varphi^{*i}\bar{\psi}^i = 0, \quad A_0 = -\frac{g^2}{2N}\bar{\psi}^{ci}\psi^i, \quad A_0^* = -\frac{g^2}{2N}\bar{\psi}^i\psi^{ci}. \tag{3.4}$$

The equations (3.4) is similar to $\mathcal{N} = 1$ supersymmetric $O(N)$ model with zero radius.

3.2 Phase structure

Now we obtain the stationary conditions of the effective potential.

Similarly to CP^{N-1} case, we integrate out the fluctuation field and put the vacuum expectation values as follows:

$$\langle M(x)\rangle = M_0, \quad \langle D(x)\rangle = D_0,$$
$$\langle F_0(x)\rangle = F_0, \quad \langle A_0(x)\rangle = A_0, \quad \langle\text{the others}\rangle = 0.$$

Then we obtain the effective potential.

$$
\begin{aligned}
\frac{V}{N} &= \int \frac{d^3k}{(2\pi)^3}\ln(k_\mu k^\mu + M_0^2 + D_0 + |A_0|^2 - |F_0|^2) - \int \frac{d^3k}{(2\pi)^3}tr\ln(-k^\mu\gamma_\mu - M_0 - |A_0|^2) \\
&\quad + \frac{1}{N}\tilde{\varphi}^{i*}(M_0^2 + D_0 + A_0 A_0^*)\tilde{\varphi}^i - \frac{1}{g^2}D_0 - \frac{1}{2N}(F_0\varphi^i\varphi^i + F_0^*\varphi^{*i}\varphi^{*i}) \\
&= -\frac{1}{6\pi}\frac{1}{2}\Big[|M_0^2 + D_0 + |A_0|^2 + |F_0||^{\frac{3}{2}} + |M_0^2 + D_0 + |A_0|^2 - |F_0||^{\frac{3}{2}}\Big] \\
&\quad + \frac{1}{6\pi}\frac{1}{2}\Big[|M_0 + |A_0||^3 + |M_0 - |A_0||^3\Big] - \Big(\frac{1}{g^2} - \frac{1}{2\pi^2}\Lambda\Big)D_0 \\
&\quad + \frac{1}{N}\Big(M_0^2\varphi^{*i}\varphi^i + A_0^*A_0\varphi^{*i}\varphi^i + D_0\varphi^{*i}\varphi^i - \frac{1}{2}(F_0\varphi^i\varphi^i + F_0^*\varphi^{*i}\varphi^{*i})\Big), \\
&= -\frac{1}{6\pi}\frac{1}{2}\Big[|M_0^2 + D_0 + |A_0|^2 + |F_0||^{\frac{3}{2}} + |M_0^2 + D_0 + |A_0|^2 - |F_0||^{\frac{3}{2}}\Big] \\
&\quad + \frac{1}{6\pi}\frac{1}{2}\Big[|M_0 + |A_0||^3 + |M_0 - |A_0||^3\Big] + \frac{m}{4\pi}D_0 \\
&\quad + \frac{1}{N}\Big(M_0^2\varphi^{*i}\varphi^i + A_0^*A_0\varphi^{*i}\varphi^i + D_0\varphi^{*i}\varphi^i - \frac{1}{2}(F_0\varphi^i\varphi^i + F_0^*\varphi^{*i}\varphi^{*i})\Big).
\end{aligned}
$$

We use the same renormalization of coupling constant and the same invariant mass as CP^{N-1} case.

If the all contents of absolute value are positive, the stationary points satisfy the following equations.

$$
\begin{aligned}
\frac{1}{N}\frac{\partial V}{\partial M_1} &= -\frac{1}{4\pi}M_1\Big[|M_1^2 + M_2^2 + D_1|^{\frac{1}{2}} + |M_1^2 + M_2^2 + D_2|^{\frac{1}{2}} + 2\sqrt{2}|M_1|\Big] \\
&\quad + \frac{1}{N}M_1(\varphi_1^2 + \varphi_2^2) = 0 \\
\frac{1}{N}\frac{\partial V}{\partial M_2} &= -\frac{1}{4\pi}M_1\Big[|M_1^2 + M_2^2 + D_1|^{\frac{1}{2}} + |M_1^2 + M_2^2 + D_2|^{\frac{1}{2}} + 2\sqrt{2}|M_2|\Big] \\
&\quad + \frac{1}{N}M_2(\varphi_1^2 + \varphi_2^2) = 0 \\
\frac{1}{N}\frac{\partial V}{\partial D} &= -\frac{1}{8\pi}\Big[|M_1^2 + M_2^2 + D_1|^{\frac{1}{2}} + |M_1^2 + M_2^2 + D_2|^{\frac{1}{2}}\frac{m}{8\pi} + \frac{1}{2N}(\varphi_1^2 + \varphi_2^2) = 0 \\
\frac{1}{N}\frac{\partial V}{\partial F_0} &= \frac{1}{16\pi}\Big[e^{-2i\theta}|M_1^2 + M_2^2 + D_1|^{\frac{1}{2}} - e^{-2i\theta}|M_1^2 + M_2^2 + D_2|^{\frac{1}{2}}\Big] - \frac{1}{2N}\varphi^i\varphi^i = 0 \\
\frac{1}{N}\frac{\partial V}{\partial F_0^*} &= \frac{1}{16\pi}\Big[e^{2i\theta}|M_1^2 + M_2^2 + D_1|^{\frac{1}{2}} - e^{2i\theta}|M_1^2 + M_2^2 + D_2|^{\frac{1}{2}}\Big] - \frac{1}{2N}\varphi^{*i}\varphi^{*i} = 0
\end{aligned}
$$

where we defined new parameters as follow:

$$
F_0 = |F_0|e^{2i\theta}, \quad \frac{1}{\sqrt{2}}(M + |A_0|) = M_1, \quad \frac{1}{\sqrt{2}}(M - |A_0|) = M_2,
$$
$$
D - |F_0| = D_1, \quad D + |F_0| = D_2, \quad \varphi^i e^{i\theta} = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2).
$$

We find three kinds of phases.

1. Chern-Simons phases

   The case of $M_0 = m$, $|A_0| = 0$, $D = F_0 = \varphi_1 = \varphi_2 = 0$.

   In this phase, both global $SO(N)$ and gauged $U(1)$ symmetries are preserved. N dynamical scalar and spinor fields obtain masses m due to the vacuum expectation value of M_0. Similarly to CP^{N-1} case, the auxiliary fields have twice masses of the dynamical fields. The auxiliary field with the vacuum expectation value M corresponds to the bound state of the fermion and anti-fermion.

   Expanding the effective action, the gauge field parts of the effective action have the Chern-Simons interaction term in the next-to-leading order of $1/N$.

   $$S_{eff} \sim \int \frac{d^3 p}{(2\pi)^3} - v^\mu(-p) \frac{N}{4\pi} (p^2 \eta_{\mu\nu} - p_\mu p_\nu - 2mi\epsilon_{\mu\nu\rho}p^\rho) I^{-1}(p) v^\mu, \tag{3.5}$$

   where,

   $$I(p) = \frac{\sqrt{-p^2}}{\arctan(\sqrt{\frac{-p^2}{4m}})}. \tag{3.6}$$

   The gauge field v^μ obtain the mass $2m$ becuase of this interaction term in Chern- Simons phase.

2. Higgs phases The case of $M_0 = 0$, $|A_0| = m$, $D = F_0 = \varphi_1 = \varphi_2 = 0$.

   In this phase, global $SO(N)$ symmetry is protected and N dynamical fields have masses due to the vacuum expectation value of A_0. However gauged $U(1)$ symmetry is broken because the superfields Φ_0, which has nonzero $U(1)$ charge, has nonvanishing vacuum expectation value.

   Expanding the effective action, we calculate all propagators of auxiliary fields in next-to-leading order of $1/N$:

   $$G^{A_R} = \frac{4\pi}{N} \frac{i}{p^2 - 4m^2} I(p),$$

   $$G^M = \frac{4\pi}{N} \frac{i}{p^2 - 4m^2} I(p),$$

   $$G^{D'} = \frac{4\pi i}{N} I(p),$$

   $$G^{F_0} = \frac{8\pi}{N} I(p),$$

   $$G^{\lambda'} = \frac{4\pi}{N} \frac{2i}{p^\mu \gamma_\mu + 2m} I(p),$$

   $$G^{\psi'_0} = \frac{4\pi}{N} \frac{2i}{p^\mu \gamma_\mu - 2m} I(p),$$

   $$G^U_{\mu\nu} = -\frac{4\pi}{N} \frac{i}{p^2 - 4m^2} (\eta_{\mu\nu} - \frac{p_\mu p_\nu}{4m^2}) I(p),$$

where we combinate the fields λ, ψ, D, A_0 and v_μ to diagonalize the two point functions as follow:

$$
\begin{aligned}
\lambda' &= \frac{1}{\sqrt{2}}(\lambda - \psi_0^c), \\
\psi_0' &= \frac{i}{\sqrt{2}}(\lambda^c + \psi_0), \\
A_0 &= m + A_R + iA_I, \\
D' &= D + 2mA_R, \\
U_\mu(p) &= v_\mu(p) - (2m)^{-1}ip_\mu A_I.(p)
\end{aligned}
$$

From these propagators, we find the auxiliary fields M and A_R has twice masses of dynamical fields. From eq. of motion for A_0, we find the fields A_0^* corresponds to the pair of fermion and fermion:

$$
A_0^* = \frac{g^2}{4N}\bar{\psi}^{ci}\psi^i. \tag{3.7}
$$

In this phase, the gauge bosons acquire masses through the Higgs mechanism and the imaginary part of A_0 is removed from the theory.

3. broken phases

The case of $M_0 = 0$, $|A_0| = 0$, $D = F_0 = 0$, $0 = \frac{4\pi}{N}\varphi_1^2 + m$, $0 = \frac{4\pi}{N}\varphi_2^2 + m$ ($m < 0$).

In this phase, both global $SO(N)$ and gauged $U(1)$ symmetries are broken. There are $N - 2$ massless Nambu-Goldstone bosons and their superpartners. Similarly to the broken phase of CP^{N-1} model, the propagators of the fields v^μ, M, λ, D, A_0, ψ_0, F_0, φ^{N-1}, φ^N, ψ^{N-1} and ψ^N do not have the singular point, shown Appendix B. We find these fields are not physical particles.

3.3 The β function

Finally, we calculate the β function in Chern-Simon phase because the β function is determined by the UV behavior and gives same result in either phase. We summarize the Feynman rule in Chern-Simon phase in Appendix A.2.

Similarly to CP^{N-1} model, the gap equation is given by

$$
\frac{N}{g^2} = \int \frac{d^3p}{(2\pi)^3}\langle\varphi^i(p)\varphi^{*i}(0)\rangle. \tag{3.8}
$$

The gap equation of the leading order of $1/N$ expansion is

$$
\frac{N}{g^2} = \int \frac{d^3p}{(2\pi)^3}\frac{iN}{p^2 - m^2}. \tag{3.9}
$$

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Then the β function of the coupling constant is given by

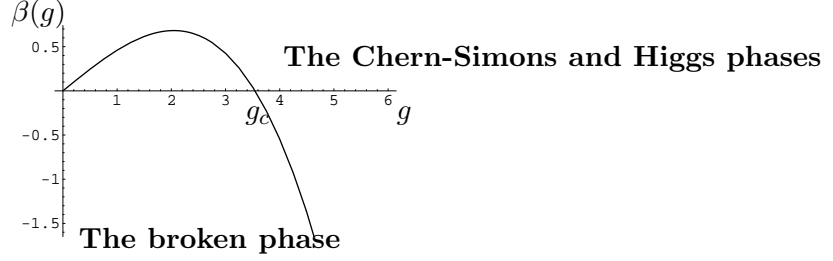$$\beta(g_R) = -\frac{1}{8\pi}g_R^3 + \frac{1}{2}g_R. \tag{3.10}$$

Figure 3: The β function of the coupling constant g for Q^{N-2} model.

In the next-to-leading order of $1/N$, we calculate eight Feynman diagrams in Fig.4. We introduce the UV cutoff Λ then the gap equation is given by

$$\begin{aligned}
\frac{1}{g^2} &= \frac{1}{2\pi^2}\left((\Lambda - \frac{\pi}{2}m) - \frac{2}{N}(\Lambda - \frac{\pi}{2}2m)\right) \\
&= \frac{\Lambda}{2\pi}(1 - \frac{2}{N}) - \frac{m}{4\pi}(1 - \frac{4}{N}).
\end{aligned} \tag{3.11}$$

In three dimensional Q^{N-2} model, the coupling constant g has the canonical dimension $\frac{1}{2}$. Then the β function of such coupling constant depends on the subtraction of finite quantity [24]. We define the renormalized coupling constant g_R by

$$\frac{1}{g^2} - (1 - \frac{2}{N})\frac{\Lambda - \mu}{2\pi^2} = \frac{\mu}{g_R^2}, \tag{3.12}$$

and obtain the next-to-leading order β function.

$$\beta = \frac{1}{2}g_R - \frac{1}{4\pi^2}(1 - \frac{2}{N})g_R^3 \tag{3.13}$$

We find this β function coincides with the WRG result and is shown in Fig.3.

Similarly to the CP^{N-1} model, the region where the coupling constant is smaller than the critical point g_c corresponds to broken phase, while the region where the coupling constant is larger than g_c corresponds to Chern-Simons and Higgs phase.

4   Conclusion

In this paper, we discuss two concrete $\mathcal{N} = 2$ supersymmetric nonlinear sigma models, CP^{N-1} and Q^{N-2} models, using $1/N$ expansion.
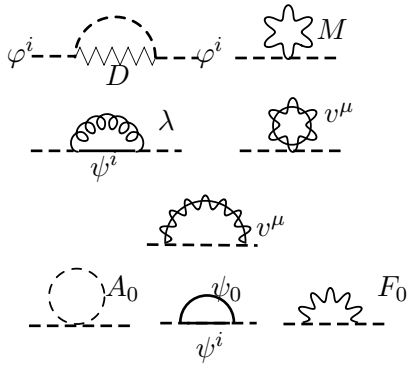
Figure 4: The Feynman diagrams contributing to the next-to-leading order correction of φ propagetor

We derived the β function and found the non-trivial UV fixed points of these two model. In the CP^{N-1} case, the β function accidentally has no next-to-leading order correction in the $1/N$, however in the case of Q^{N-2} model, there is a next-to-leading order correction. We found these β functions up to next-to-leading order of $1/N$ expansion coincide with the ones obtained by using the WRG analysis.

Furthermore, we investigated the phase structures of both CP^{N-1} and Q^{N-2} models.

The CP^{N-1} model has two phases: the $SU(N)$ symmetric and broken phases. In the symmetric phase, all dynamical fields have mass (m) due to the vacuum expectation value of the auxiliary field M. The auxiliary fields also have a mass $(2m)$, and the field M corresponds to the bound state of two dynamical spinor field $\bar{\psi}^i \psi^i$. On the other hand, in the broken phase, a dynamical field has a vacuum expectation value, and global $SU(N)$ symmetry is broken. There are $(N-1)$ massless Nambu-Goldstone bosons and their superpartners, and non of the auxiliary fields are physical particles.

The Q^{N-2} model has three phases: the Chern-Simons, Higgs and $SO(N)$ broken phases. In the Chern-Simons and Higgs phase, all dynamical fields have a mass (m) due to the vacuum expectation value of the auxiliary field M and A_0, respectively. The auxiliary fields also have a mass $(2m)$, and the field M corresponds to the bound state of the fermion and anti-fermion; $M \sim \bar{\psi}^i \psi^i$, and A_0 corresponds to the bound state of two fermions $A_0 \sim \bar{\psi}^{ci} \psi^i$. In the Chern-Simons phase, the gauge fields obtain the mass due to the induced Chern-Simons interaction. In the Higgs phase, the gauge field acquire the mass through the Higgs mechanism and the imaginary part of the field A_0 is absorbed by the gauge field. In the broken phase, two components of dynamical fields φ^{N-1} and φ^N have the vacuum expectation values, then global $SO(N)$ symmetry is broken down to $SO(N-2)$. There are $(N-2)$ massless Nambu-Goldstone bosons and their superpartners, and non of the auxiliary fields are physical particles.

Acknowledgements

A The Feynman rules

We summarize the Feynman rules of $\mathbf{C}P^{N-1}$ and Q^{N-2} models in symmetric phase.

A.1 The $\mathbf{C}P^{N-1}$ model

ψ^i	——	$\bar{\psi}^j$	$\dfrac{i\delta_{ij}}{\not{p}-m}$		
φ^i	- - - -	φ^{*j}	$i\dfrac{\delta_{ij}}{p^2-m^2}$		
M			D_M		
v^μ			$D_{\mu\nu}$		
D			D_D		
λ			D_λ		

$-(p_2-p_1)^\mu \delta^{ij}$ $-i\gamma^\mu \delta_{ij}$

$-i\delta_{ij}$ $-i\delta_{ij}$

$i\eta^{\mu\nu}\delta_{ij}$ $-\delta_{ij}$

$-2i\delta_{ij}$ δ_{ij}
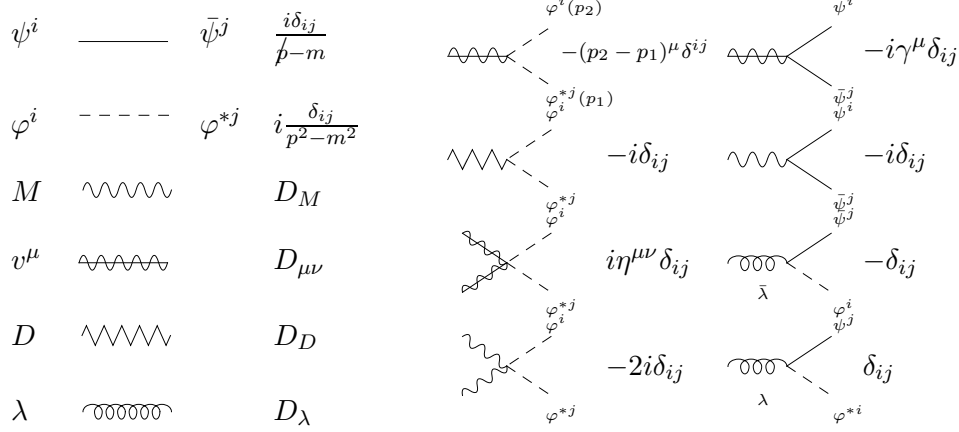
Figure 5: The Feynman rules of CP^{N-1} model in Minkowski spaces

Here,

$$D_M = \frac{4\pi i}{N}\frac{1}{p^2-4m^2}I(p^2),$$

$$D_{\mu\nu} = \frac{4\pi}{N}\frac{1}{p^2-4m^2}[-i\eta_{\mu\nu}+(1+\alpha+\frac{4m^2\alpha}{p^2})\frac{ip_\mu p_\nu}{p^2}+\frac{2m\epsilon_{\mu\nu\lambda}p^\lambda}{p^2}]I(p^2),$$

$$D_D = \frac{4\pi i}{N}I(p^2),$$

$$D_\lambda = -\frac{8\pi i}{N}\frac{1}{\not{p}+2m}I(p^2),$$

$$I(p) = \frac{\sqrt{-p^2}}{\arctan(\sqrt{\frac{-p^2}{4m}})}.$$

ψ^i —— $\bar\psi^j$ $\quad \dfrac{i\delta_{ij}}{p^\mu\gamma_\mu - m}$

φ^i ----- φ^{*j} $\quad i\dfrac{\delta_{ij}}{p^2 - m^2}$

M $\quad$ D_M

v^μ $\quad$ $D_{\mu\nu}$

D $\quad$ D_D

ξ $\quad$ D_ξ

A_0 $\quad$ D_{A_0}

ψ_0 $\quad$ D_{ψ_0}

F_0 $\quad$ D_{F_0}

Vertices:

$\varphi^i(p_2)$, v^μ, $\varphi^{*j}(p_1)$: $-(p_2 - p_1)^\mu \delta^{ij}$

$\bar\psi^j$, $\bar\lambda$, φ^i : $-\delta_{ij}$

φ^i, D, φ^{*j} : $-i\delta_{ij}$

λ, φ^{*i}, ψ^j : δ_{ij}

v^μ, φ^i, v^ν, φ^{*j} : $i\eta^{\mu\nu}\delta_{ij}$

φ^i, $\bar\psi_0$, ψ^j : $-\delta_{ij}$

M, φ^i, M, φ^{*j} : $-2i\delta_{ij}$

φ^{*i}, $\bar\psi^j$: $-\delta_{ij}$

F_0, φ^j, φ^{*i} : $i\delta_{ij}$

A_0, ψ_0, φ^i : $-i\delta_{ij}$

F_0^*, φ^{*j} : $i\delta_{ij}$
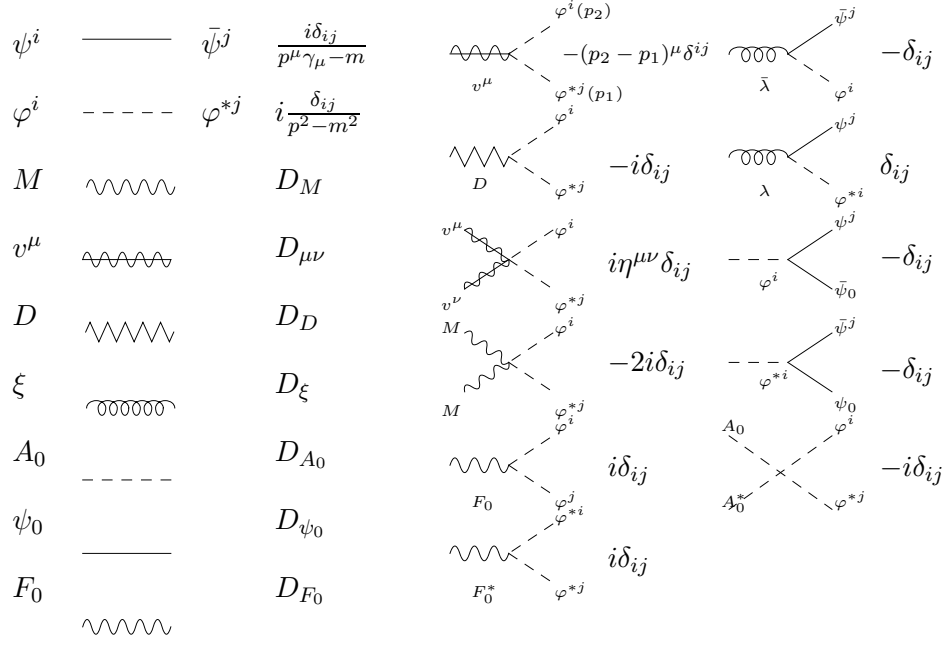
A_0^*, φ^{*j} : $-i\delta_{ij}$

Figure 6: The Feynman rule of Q^{N-2} model in Minkowski spaces. We show only the parts which are concerned the next-to-leading correction of φ^i propagator.

Here,

$$D_M = \frac{4\pi i}{N}\frac{1}{p^2 - 4m^2}I(p^2),$$

$$D_{\mu\nu} = \frac{4\pi}{N}\frac{1}{p^2 - 4m^2}\left[-i\eta_{\mu\nu} + (1 + \alpha + \frac{4m^2\alpha}{p^2})\frac{ip_\mu p_\nu}{p^2} + \frac{2m\epsilon_{\mu\nu\lambda}p^\lambda}{p^2}\right]I(p^2),$$

$$D_D = \frac{4\pi i}{N}I(p^2),$$

$$D_\xi = -\frac{8\pi i}{N}\frac{1}{p^\mu\gamma_\mu + 2m}I(p^2),$$

$$D_{A_0} = \frac{8\pi i}{N}\frac{1}{p^2 - 4m^2}I(p^2),$$

$$D_{\psi_0} = \frac{8\pi i}{N}\frac{1}{p^\mu\gamma_\mu + 2m}I(p^2)$$

$$D_{F_0} = \frac{8\pi i}{N}I(p^2).$$

B The broken phase of Q^{N-2} model

From the stationaly condition of the effective potential, we obtain the set of vacuum expectation values,

$$M_0 = 0, \ |A_0| = 0, \ |D_1|^{\frac{1}{2}} = \frac{4\pi}{N}\varphi_1^2 + m, \ |D_2|^{\frac{1}{2}} = \frac{4\pi}{N}\varphi_2^2 + m. \tag{B.1}$$

Then the effective potential becomes

$$V_{eff} = \frac{1}{24\pi}\left[|\frac{4\pi}{N}\varphi_1^2 + m|^3 + |\frac{4\pi}{N}\varphi_2^2 + m|^3\right], \tag{B.2}$$

and the minimal value of this effective potential is realized when

$$\varphi_1^2 = \frac{N}{4\pi}|m|, \quad \varphi_2^2 = \frac{N}{4\pi}|m|. \tag{B.3}$$

Using $SO(N)$ symmetry and the constraint $\varphi_1\varphi_2 = 0$ which is obtained by the stationary condition, we choose the vacuum expectation values as follow:

$$\varphi_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ i\sqrt{\frac{N}{4\pi}|m|} \\ \sqrt{\frac{N}{4\pi}|m|} \end{pmatrix}. \tag{B.4}$$

Thus global $SO(N)$ symmetry breaks to $SO(N-2)$.

Futhermore, we put the fluctuation rewritten the φ^{N-1} and φ^N, and these superpartnars ψ^{N-1} and ψ^N as

$$\frac{1}{\sqrt{2}}(\varphi^N - i\varphi^{N-1}) = \mathcal{A}_N,$$

$$\frac{1}{\sqrt{2}}(\varphi^N + i\varphi^{N-1}) = \mathcal{A}_{N-1},$$

$$\frac{1}{\sqrt{2}}(\psi^N - i\psi^{N-1}) = \Psi_N,$$

$$\frac{1}{\sqrt{2}}(\psi^N + i\psi^{N-1}) = \Psi_N.$$

Then the effective action expanding up to the next-to-leading order of $1/N$ can be written by

$$
\begin{aligned}
S_{eff} = \int \frac{d^3p}{(2\pi)^3} \Bigg[& \Big\{ A_0^*(-p)(-\frac{N}{16}\sqrt{-p^2} - \frac{N}{4\pi}|m|)A_0(p) + \frac{1}{2}M(-p)(-\frac{N}{8}\sqrt{-p^2} - \frac{N}{2\pi}|m|)M(p) \\
& - \frac{1}{2}v^\mu(-p)\left(\frac{N}{8}\frac{p^2\eta_{\mu\nu} - p_\mu p_\nu}{\sqrt{-p^2}} - \frac{1}{2\pi}|m| \right)v^\nu(p) + \bar{\lambda}(-p)\frac{N}{16}\frac{p^\mu\gamma_\mu}{\sqrt{-p^2}}\lambda(p) \\
& + \bar{\psi}_0(-p)\frac{N}{16}\frac{p^\mu\gamma_\mu}{\sqrt{-p^2}}\psi_0(p) + \frac{1}{2}D(-p)\frac{N}{8\sqrt{-p^2}}D(p) + F_0^*(-p)\frac{N}{16\sqrt{-p^2}}F_0(p) \Big\} \\
& + \sum_{k=1}^{N-2} A_k^*(-p)p^2 A_k(p) + \mathcal{A}_{N-1}^*(-p)p^2\mathcal{A}_{N-1}(p) + \mathcal{A}_N^*(-p)p^2\mathcal{A}_N(p) \\
& + \sqrt{\frac{N}{4\pi}}|m|\left(D(-p)\mathcal{A}_N(p) + D(-p)\mathcal{A}_N^*(p)\right) + \sqrt{\frac{N}{4\pi}}|m|\left\{ F_0(-p)\mathcal{A}_{N-1}(p) + F_0^*(-p)\mathcal{A}_{N-1}^*(p) \right\} \\
& + \sum_{k=1}^{N-2} \bar{\psi}_k(-p)p^\mu\gamma_\mu\psi_k(p) + \bar{\Psi}_{N-1}(-p)p^\mu\gamma_\mu\Psi_{N-1}(p) + \bar{\Psi}_N(-p)p^\mu\gamma_\mu\Psi_N(p) \\
& - i\sqrt{\frac{N}{4\pi}}|m|\left(\bar{\Psi}_N(-p)\bar{\lambda}(p) - \Psi_N(-p)\lambda(p) \right) - \sqrt{\frac{N}{4\pi}}|m|\left(\bar{\psi}_0(-p)\bar{\Psi}_{N-1}(p) + \psi_0(-p)\Psi_{N-1}(p) \right) \\
& - \sqrt{\frac{N}{4\pi}}|m|v^\mu(-p)p_\mu\left(\mathcal{A}_N(p) - \mathcal{A}_N^*(p) \right) + (\text{interactions}) \Bigg] .
\end{aligned}
\tag{B.5}
$$

This effective action is similar to the one of CP^{N-1} broken phase. We put $\mathcal{A}_N = \frac{1}{\sqrt{2}}(\mathcal{A}_N^R + i\mathcal{A}_N^I)$ and rewrite the effective action as

$$
S_{eff} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} (D \ \mathcal{A}_N^R)
\begin{pmatrix} \frac{N}{8\sqrt{-p^2}} & \sqrt{\frac{N}{2\pi}|m|} \\ \sqrt{\frac{N}{2\pi}|m|} & p^2 \end{pmatrix}
\begin{pmatrix} D \\ \mathcal{A}_N^R \end{pmatrix}
\tag{B.6}
$$

$$
+ (F_0 \ \mathcal{A}_{N-1}^*)
\begin{pmatrix} \frac{N}{16\sqrt{-p^2}} & \sqrt{\frac{N}{4\pi}|m|} \\ \sqrt{\frac{N}{4\pi}|m|} & p^2 \end{pmatrix}
\begin{pmatrix} F_0 \\ \mathcal{A}_{N-1} \end{pmatrix}
\tag{B.7}
$$

$$
+ (\bar{\lambda} \ \Psi_N)
\begin{pmatrix} \frac{Np^\mu\gamma_\mu}{16\sqrt{-p^2}} & -i\sqrt{\frac{N}{4\pi}|m|} \\ i\sqrt{\frac{N}{4\pi}|m|} & p^\mu\gamma_\mu \end{pmatrix}
\begin{pmatrix} \lambda \\ \bar{\Psi}_N \end{pmatrix}
\tag{B.8}
$$

$$
+ (\bar{\psi}_0 \ \Psi_{N-1})
\begin{pmatrix} \frac{Np^\mu\gamma_\mu}{16\sqrt{-p^2}} & -\sqrt{\frac{N}{4\pi}|m|} \\ -\sqrt{\frac{N}{4\pi}|m|} & p^\mu\gamma_\mu \end{pmatrix}
\begin{pmatrix} \psi_0 \\ \bar{\Psi}_{N-1} \end{pmatrix} \cdots .
\tag{B.9}
$$

The eigen value of these matrix has no solution, then we find these fields are not physical in this phase.

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